

Products of radial derivative and integral-type operators from Zygmund spaces to Bloch spaces

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Abstract. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on the unit ball $\mathbb{B} \in \mathbb{C}^n$. In this paper we investigate the boundedness and compactness of the products of radial derivative operator and the following integral-type operator

$$I_{\varphi}^g f(z) = \int_0^1 \Re f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad z \in \mathbb{B}$$

where $g \in H(\mathbb{B})$, $g(0) = 0$, φ is a holomorphic self-map of \mathbb{B} , between Zygmund spaces and Bloch spaces.

Keywords: radial derivative operator; integral-type operator; Zygmund space; Bloch space

1. Introduction

Let $H(\mathbb{B})$ denote the space of all holomorphic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in the complex vector space \mathbb{C}^n and $\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n$. Let

$$\Re f(z) = \sum z_j \frac{\partial f}{\partial z_j}(z)$$

stand for the radial derivative of $f \in H(\mathbb{B})[1]$. It is easy to see that, if $f \in H(\mathbb{B})$, $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$, where α is a multi-index, then $\Re f(z) = \sum_{\alpha} |\alpha| a_{\alpha} z^{\alpha}$. We write $\Re^m f = \Re(\Re^{m-1} f)$.

The Bloch space $\mathcal{B}(\mathbb{B})$ is the space of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2) |\nabla f(z)| < \infty,$$

The little Bloch space $\mathcal{B}_0(\mathbb{B})$ is the space of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\nabla f(z)| = 0.$$

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It is well known that $f \in \mathcal{B}(\mathbb{B})$ if and only if

$$b(f) := \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re f(z)| < \infty,$$

and that $f \in \mathcal{B}_0$ if and only if $\lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re f(z)| = 0$. Moreover, the following asymptotic relation holds[2]:

$$\|f\|_{\mathcal{B}} \asymp |f(0)| + b(f).$$

Let \mathcal{Z} denote the class of all $f \in H(\mathbb{B})$, such that

$$\sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2 f(z)| < \infty. \quad (1)$$

Therefore, \mathcal{Z} is called the Zygmund class. Under the natural norm

$$\|f\|_{\mathcal{Z}} := |f(0)| + |f'(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2 f(z)| < \infty. \quad (2)$$

\mathcal{Z} becomes a Banach space. Zygmund class with this norm will be called the Zygmund space.

The little Zygmund space \mathcal{Z}_0 denote the closure in \mathcal{Z} of the set of all polynomials. From Theorem 7.2 of[3], we see that

$$f \in \mathcal{Z}_0 \Leftrightarrow \lim_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2 f(z)| = 0. \quad (3)$$

Suppose that $g \in H(\mathbb{B})$, $g(0) = 0$ and φ is a holomorphic self-map of \mathbb{B} , then an integral-type operator, denote by I_{φ}^g on $H(\mathbb{B})$, is defined as follows:

$$I_{\varphi}^g f(z) = \int_0^1 \Re f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad g \in H(\mathbb{B}), \quad z \in \mathbb{B} \quad (4)$$

Operator (4) is related to the following operators

$$T_g(f)(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad I_g(f) = \int_0^1 \Re f(tz) g(tz) \frac{dt}{t}. \quad (5)$$

acting on $H(\mathbb{B})$, introduced in [4] and studied in [5-10], as well as the operator T_g introduced in [11] acting on holomorphic functions on the unit polydisc (see, also [12], [13], as well as [14] for a particular case of the operator). One of motivations for introducing operator I_{φ}^g stems from the operator introduced in [15]. Some characterizations of the boundedness and compactness of these and some other integral-type operators mostly in \mathbb{C}^n , can be found, for example, in [4, 6, 7-9, 15-31].

In this paper we study the boundedness and compactness of products of \Re and I_φ^g between Zygmund space and Bloch spaces on the unit Ball.

Throughout this paper, constants are denoted by C , they are positive and may differ from one occurrence to the other.

2. Auxiliary results

Lemma 1^[19]. Let \Re be the radial derivative operator. The product of \Re and I_φ^g

$$\Re[I_\varphi^g(f)](z) = \Re f(\varphi(z))g(z) \quad (6)$$

Lemma 2^[17]. Suppose $f \in \mathcal{Z}$. The following statements are true.

(a). There is a positive constant C independent of f such that

$$|\Re f(z)| \leq C \|f\|_{\mathcal{Z}} \ln \frac{e}{1 - |z|^2}. \quad (7)$$

(b). There is a positive constant C independent of f such that

$$\|f\|_\infty = \sup_{|z| < 1} |f(z)| \leq C \|f\|_{\mathcal{Z}}. \quad (8)$$

For studying the compactness of the operator $\Re I_\varphi^g : \mathcal{Z} \rightarrow \mathcal{B}$, we need the following Lemma. The proof of the lemma is standard, hence we omit the details.

Lemma 3. Assume that $g \in H(\mathbb{B})$, φ be a holomorphic self-map of \mathbb{B} . Let $T = \Re I_\varphi^g$, then $T : \mathcal{Z}(\text{or } \mathcal{Z}_0) \rightarrow \mathcal{B}$ is compact if and only if T is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $\mathcal{Z}(\text{or } \mathcal{Z}_0)$ which converges to 0 uniformly on compact subsets of \mathbb{B} , $Tf_k \rightarrow 0$ as $k \rightarrow \infty$.

Lemma 4^[17]. A closed set K in \mathcal{B}_0 is compact if and only if it is bounded and satisfies

$$\lim_{|z| \rightarrow 1} \sup_{f \in K} (1 - |z|^2) |\Re f(z)| = 0. \quad (9)$$

3. The boundedness and compactness of $\Re I_\varphi^g : \mathcal{Z}(\mathcal{Z}_0) \rightarrow \mathcal{B}(\mathcal{B}_0)$

Theorem 1. Let φ be a holomorphic self-map of \mathbb{B} . Then the following statements are equivalent.

- (a) $\Re I_\varphi^g : \mathcal{Z} \rightarrow \mathcal{B}$ is bounded;
- (b) $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is bounded;
- (c)

$$\sup_{z \in \mathbb{B}} \frac{(1 - |z|^2) |\Re \varphi(z)| |g(z)|}{1 - |\varphi(z)|^2} < \infty, \quad (10)$$

and

$$\sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} < \infty. \quad (11)$$

Proof. (a) \Rightarrow (b) This implication is obvious.

(b) \Rightarrow (c) Assume that $\Re I_\varphi^g: \mathcal{Z}_0 \rightarrow \mathcal{B}$ is boundedness, i.e., there exists a constant C such that

$$\|\Re I_\varphi^g(f)\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{Z}},$$

for all $f \in \mathcal{Z}_0$. Taking the functions $f_j(z) = z_j \in \mathcal{Z}_0$ and $f_j(z) = z_j - z_j^2 \in \mathcal{Z}_0, j = 1, 2, \dots, n$, we get

$$\sup_{z \in \mathbb{B}} (1 - |z|^2) |\varphi_j(z)| |\Re \varphi(z) g(z) + \Re g(z)| < \infty, \quad (12)$$

and

$$\sup_{z \in \mathbb{B}} (1 - |z|^2) |(\varphi_j(z) - 4\varphi_j^2(z))(\Re \varphi(z) g(z) + \Re g(z)) + 2\varphi_j^2(z) \Re g(z)| < \infty. \quad (13)$$

Using (12) and the boundedness of functions φ_j , we have that

$$\sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re \varphi(z) g(z) + \Re g(z)| < \infty. \quad (14)$$

Then with (13), (14) and the boundedness of functions φ_j , we have that

$$\sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re g(z)| < \infty, \quad \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re \varphi(z) g(z)| < \infty \quad (15)$$

Set

$$h(\zeta) = (\zeta - 1) \left[\left(1 + \ln \frac{1}{1 - \zeta} \right)^2 + 1 \right], \zeta \in \mathbb{C},$$

and

$$h_a(z) = \frac{h(< z, a >)}{|a|^2} \left(\ln \frac{1}{1 - |a|^2} \right)^{-1},$$

for $a \in \mathbb{B} \setminus \{0\}$. It is known that $h_a(z) \in \mathcal{Z}_0$ (see [17]). Since

$$\Re h_a(z) = \frac{< z, a >}{|a|^2} \left(\ln \frac{1}{1 - < z, a >} \right)^2 \left(\ln \frac{1}{1 - |a|^2} \right)^{-1},$$

and

$$\Re^2 h_a(z) = \Re h_a(z) + \frac{2 < z, a >^2}{|a|^2 (1 - < z, a >)} \left(\ln \frac{1}{1 - < z, a >} \right) \left(\ln \frac{1}{1 - |a|^2} \right)^{-1},$$

for $|\varphi(z)| > \sqrt{1 - 1/e}$ we have

$$\begin{aligned} C \|\Re I_\varphi^g\|_{\geq} \|\Re I_\varphi^g(h_{\varphi(z)})\|_{\mathcal{B}} &\geq (1 - |z|^2) \ln \frac{1}{1 - |\varphi(z)|^2} |\Re g(z)| \\ &- \frac{2(1 - |z|^2)}{1 - |\varphi(z)|^2} |\varphi(z)|^2 |\Re \varphi(z)| |g(z)| \\ &- (1 - |z|^2) \ln \frac{1}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)|. \end{aligned}$$

Hence

$$\begin{aligned}
(1 - |z|^2) \ln \frac{1}{1 - |\varphi(z)|^2} |\Re g(z)| &\leq C + \frac{2(1 - |z|^2)}{1 - |\varphi(z)|^2} |\varphi(z)|^2 |\Re \varphi(z)| |g(z)| \\
&+ (1 - |z|^2) \ln \frac{1}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)| \\
&\leq C + (2 + e) \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)|, \quad (16)
\end{aligned}$$

which using the fact of $(1 - |\varphi(z)|^2) \ln \frac{1}{1 - |\varphi(z)|^2} \leq e$.

For $|a| > \sqrt{1 - 1/e}$, set

$$f_a(z) = \frac{h(< z, a >)}{|a|^2} (\ln \frac{1}{1 - |a|^2})^{-1} - \int_0^1 \ln \frac{1}{1 - < tz, a >} \frac{dt}{t}.$$

Then $f_a \in \mathcal{Z}_0$. It is easy to see that

$$\begin{aligned}
\Re f_a(z) &= \frac{< z, a >}{|z|^2} (\ln \frac{1}{1 - < z, a >})^2 (\ln \frac{1}{1 - |a|^2})^{-1} - \ln \frac{1}{1 - < z, a >}, \\
\Re^2 f_a(z) &= \Re f_a(z) + \frac{2 < z, a >^2}{|a|^2 (1 - < z, a >)} (\ln \frac{1}{1 - < z, a >}) (\ln \frac{1}{1 - |a|^2})^{-1} \\
&- \frac{< z, a >}{1 - < z, a >} + \ln \frac{1}{1 - < z, a >}.
\end{aligned}$$

Therefore

$$\begin{aligned}
C \|\Re I_\varphi^g\| &\geq \|\Re I_\varphi^g(f_{\varphi(z)})\|_{\mathcal{B}} = \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2(I_\varphi^g f_{\varphi(z)})(z)| \\
&= (1 - |z|^2) \left(\frac{|\varphi(z)|^2}{1 - |\varphi(z)|^2} + \ln \frac{1}{1 - |\varphi(z)|^2} \right) |\Re \varphi(z)| |g(z)| \\
&\geq (1 - |z|^2) \left(\frac{|\varphi(z)|^2}{1 - |\varphi(z)|^2} + 1 \right) |\Re \varphi(z)| |g(z)| \\
&= \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)|. \quad (17)
\end{aligned}$$

On the other hand, from (15) we have that

$$\sup_{|\varphi(z)| \leq \sqrt{1 - 1/e}} (1 - |z|^2) |\Re g(z)| \ln \frac{1}{1 - |\varphi(z)|^2} \leq \sup_{|\varphi(z)| \leq \sqrt{1 - 1/e}} (1 - |z|^2) |\Re g(z)| < \infty. \quad (18)$$

Hence from (15), (16), (17) and (18), we obtain (11). Further, from (17), we have

$$\sup_{|\varphi(z)| > \sqrt{1 - 1/e}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)| \leq C. \quad (19)$$

On the other hand, from (15) we have that

$$\sup_{|\varphi(z)| \leq \sqrt{1-1/e}} \frac{1-|z|^2}{1-|\varphi(z)|^2} |\Re \varphi(z)| |g(z)| \leq e, \quad \sup_{|\varphi(z)| \leq \sqrt{1-1/e}} (1-|z|^2) |\Re \varphi(z)| |g(z)| < \infty. \quad (20)$$

Combining (19) and (20), (10) follows.

Theorem 2. Let φ be a holomorphic self-map of \mathbb{B} . Then the following statements are equivalent.

- (a) $\Re I_\varphi^g : \mathcal{Z} \rightarrow \mathcal{B}$ is compact;
- (b) $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is compact;
- (c) $\Re I_\varphi^g : \mathcal{Z} \rightarrow \mathcal{B}$ is bounded,

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} |\Re \varphi(z)| |g(z)| = 0, \quad (21)$$

and

$$\lim_{|\varphi(z)| \rightarrow 1} (1-|z|^2) |\Re g(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0. \quad (22)$$

Proof. (a) \Rightarrow (b) This is obvious.

(b) \Rightarrow (c) Assume that $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is compact, then it is clear that $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is bounded. By theorem 1, we know that $\Re I_\varphi^g : \mathcal{Z} \rightarrow \mathcal{B}$ is bounded. Let $(z^k)_{k \in \mathbb{N}}$ be a sequence in \mathbb{B} such that $|\varphi(z^k)| \rightarrow 1$ as $k \rightarrow \infty$ and $\varphi(z^k) \neq 0, k \in \mathbb{N}$. Set

$$h_k(z) = \frac{h(< z, \varphi(z^k) >)}{|\varphi(z^k)|^2} \left(\ln \frac{1}{1-|\varphi(z^k)|^2} \right)^{-1}, k \in \mathbb{N}.$$

Then from the proof of theorem 1, we see that $h_k \in \mathcal{Z}_0$, for each $k \in \mathbb{N}$. Moreover $h_k \rightarrow 0$ uniformly on compact subsets of \mathbb{B} , as $k \rightarrow \infty$.

Since $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is compact, by Lemma 3

$$\lim_{k \rightarrow \infty} \|\Re[I_\varphi^g(h_k)]\|_{\mathcal{B}} = 0.$$

On the other hand, similar to the proof of Theorem 1, we have

$$\begin{aligned} 0 \leftarrow \|\Re I_\varphi^g(h_k)\|_{\mathcal{B}} &\geq (1-|z^k|^2) \ln \frac{1}{1-|\varphi(z^k)|^2} |\Re g(z^k)| \\ &\quad - \frac{2(1-|z^k|^2)}{1-|\varphi(z^k)|^2} |\varphi(z^k)|^2 |\Re \varphi(z^k)| |g(z^k)| \\ &\quad - (1-|z^k|^2) \ln \frac{1}{1-|\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| \\ &= (1-|z^k|^2) \ln \frac{1}{1-|\varphi(z^k)|^2} |\Re g(z^k)| \\ &\quad - M_1 \frac{(1-|z^k|^2)}{1-|\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)|, \end{aligned}$$

where $M_1 := 2|\varphi(z^k)|^2 - (1 - |\varphi(z^k)|^2) \ln \frac{1}{1-|\varphi(z^k)|^2}$.

From this we obtain

$$\lim_{k \rightarrow \infty} (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re g(z^k)| = \lim_{k \rightarrow \infty} \frac{(1 - |z^k|^2)}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| = 0, \quad (23)$$

if one of these two limits exists, which use the case of

$$\lim_{k \rightarrow \infty} [2|\varphi(z^k)|^2 + (1 - |\varphi(z^k)|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2}] = 2.$$

Next, set

$$\begin{aligned} f_k(z) &= \frac{h(< z, \varphi(z^k) >)}{|\varphi(z^k)|^2} (\ln \frac{1}{1 - |\varphi(z^k)|^2})^{-1} \\ &\quad - \int_0^1 \ln^3 \frac{1}{1 - < tz, \varphi(z^k) >} \frac{dt}{t} (\ln \frac{1}{1 - |\varphi(z^k)|^2})^{-2}. \end{aligned}$$

Since $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is compact, we have $\|\Re I_\varphi^g(f_k)\|_{\mathcal{B}} \rightarrow 0$ as $k \rightarrow \infty$. Thus, similar to the proof of Theorem 1, when $\sqrt{1 - \frac{1}{e}} < |\varphi(z^k)| < 1$

$$\begin{aligned} 0 \leftarrow \|\Re I_\varphi^g(f_k)\|_{\mathcal{B}} &\geq (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} - \frac{|\varphi(z^k)|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| \\ &\geq (1 - |z^k|^2) \frac{|\varphi(z^k)|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| \\ &\quad - (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| \\ &\geq (1 - \frac{1}{e}) \frac{1 - |z^k|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| \\ &\quad - \frac{1 - |z^k|^2}{1 - |\varphi(z^k)|^2} (1 - |\varphi(z^k)|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| \\ &= M_2 \frac{1 - |z^k|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)|, \end{aligned}$$

where $M_2 := 1 - \frac{1}{e} - (1 - |\varphi(z^k)|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2}$.

Hence

$$\lim_{k \rightarrow \infty} \frac{1 - |z^k|^2}{1 - |\varphi(z^k)|^2} |\Re \varphi(z^k)| |g(z^k)| = \lim_{k \rightarrow \infty} (1 - |z^k|^2) \ln \frac{1}{1 - |\varphi(z^k)|^2} |\Re g(z^k)| = 0. \quad (24)$$

From (24) easily following that $\lim_{k \rightarrow \infty} (1 - |z^k|^2) |\Re g(z^k)| = 0$, which altogether imply (21) and (22).

(c) \Rightarrow (a)

$$C_1 = (1 - |z|^2) |\Re \varphi(z)| |g(z)| < \infty, \quad C_2 = (1 - |z|^2) |\Re g(z)| < \infty. \quad (25)$$

For every $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)| < \varepsilon, \quad (1 - |z|^2) |\Re g(z)| \ln \frac{e}{1 - |z|^2} < \varepsilon. \quad (26)$$

Assume that $(f_k)_{k \in \mathbb{N}}$ is a sequence in \mathcal{Z} such that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{Z}} \leq L$ and f_k converges to 0 uniformly on compact subsets of \mathbb{B} as $k \rightarrow \infty$. Let $K = \{z \in \mathbb{B} : |\varphi(z)| \leq \delta\}$. Then by (25) and (26), we have that

$$\begin{aligned} \sup_{z \in \mathbb{B}} (1 - |z|^2) |(\Re^2(I_\varphi^g(f_k)))(z)| &= \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2 f_k(\varphi(z)) \Re \varphi(z) g(z) + \Re f_k(\varphi(z)) \Re g(z)| \\ &\leq \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re^2 f_k(\varphi(z)) \Re \varphi(z) g(z)| \\ &\quad + \sup_{z \in \mathbb{B}} (1 - |z|^2) |\Re f_k(\varphi(z)) \Re g(z)| \\ &\leq \sup_{z \in K} (1 - |z|^2) |\Re^2 f_k(\varphi(z)) \Re \varphi(z) g(z)| \\ &\quad + \sup_{z \in K} (1 - |z|^2) |\Re f_k(\varphi(z)) \Re g(z)| \\ &\quad + \sup_{z \in \mathbb{B} \setminus K} (1 - |z|^2) |\Re^2 f_k(\varphi(z)) \Re \varphi(z) g(z)| \\ &\quad + \sup_{z \in \mathbb{B} \setminus K} (1 - |z|^2) |\Re f_k(\varphi(z)) \Re g(z)| \\ &\leq \sup_{z \in K} (1 - |z|^2) |\Re^2 f_k(\varphi(z)) \Re \varphi(z) g(z)| \\ &\quad + \sup_{z \in K} (1 - |z|^2) |\Re f_k(\varphi(z)) \Re g(z)| \\ &\quad + \sup_{z \in \mathbb{B} \setminus K} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z) g(z)| \|f_k\|_{\mathcal{Z}} \\ &\quad + C \sup_{z \in \mathbb{B} \setminus K} \ln \frac{e}{1 - |\varphi(z)|^2} |\Re g(z)| \|f_k\|_{\mathcal{Z}} \\ &\leq C_1 \sup_{z \in K} |\Re^2 f_k(\varphi(z))| + C_2 \sup_{z \in K} |\Re f_k(\varphi(z))| + (C + 1) \varepsilon \|f_k\|_{\mathcal{Z}} \end{aligned}$$

Hence

$$\begin{aligned} \|\Re I_\varphi^g(f_k)\|_{\mathcal{B}} &\leq C_1 \sup_{z \in K} |\Re^2 f_k(\varphi(z))| + C_2 \sup_{z \in K} |\Re f_k(\varphi(z))| \\ &\quad + (C + 1) \varepsilon \|f_k\|_{\mathcal{Z}} + |f'_k(\varphi(0))| |\varphi'(0)| \end{aligned}$$

Since $(f_k)_{k \in \mathbb{N}}$ converges to 0 uniformly on compact subsets of \mathbb{B} as $k \rightarrow \infty$, Cauchy's estimate gives that $\Re f_k \rightarrow 0$ and $\Re^2 f_k \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of \mathbb{B} . Hence, letting $k \rightarrow \infty$, we obtain

$$\lim_{k \rightarrow \infty} \|\Re I_\varphi^g(f_k)\|_{\mathcal{B}} = 0.$$

Theorem 4. Let φ be a holomorphic self-map of \mathbb{B} . Then $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}_0$ is bounded if and only if $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is bounded

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re g(z)| = 0, \quad (27)$$

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re \varphi(z)| |g(z)| = 0. \quad (28)$$

Proof: Assume that $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}_0$ is bounded. Then, it is clear that $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is bounded. Taking the function $f_j(z) = z_j$ and $f_j(z) = z_j - z_j^2, j = 1, 2, \dots, n$, we obtain (27),(28).

Assume that $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is bounded and (27),(28) holds. Then for each polynomial p , we have that

$$\begin{aligned} (1 - |z|^2) |\Re^2(I_\varphi^g p)(z)| &\leq (1 - |z|^2) |\Re^2 p(\varphi(z))| |\Re \varphi(z)| |g(z)| \\ &+ (1 - |z|^2) |\Re p(\varphi(z))| |\Re g(z)|, \end{aligned} \quad (29)$$

from (27),(28) it follows that $\Re I_\varphi^g p \in \mathcal{B}_0$. Since the set of all polynomials is dense in \mathcal{Z}_0 , we have that for every $f \in \mathcal{Z}_0$, there is a sequence of polynomials $(p_n)_{n \in \mathbb{N}}$ such that $\|f - p_n\|_{\mathcal{Z}} \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\|\Re I_\varphi^g(f) - \Re I_\varphi^g(p_n)\|_{\mathcal{B}} \leq \|\Re I_\varphi^g\|_{\mathcal{Z}_0 \rightarrow \mathcal{B}} \|f - p_n\|_{\mathcal{Z}} \rightarrow 0$$

as $n \rightarrow \infty$. Since the operator $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is bounded, hence $\Re I_\varphi^g(\mathcal{Z}_0) \subseteq \mathcal{B}_0$.

Theorem 5. Let φ be a holomorphic self-map of \mathbb{B} . Then the following statements are equivalent.

- (a) $\Re I_\varphi^g : \mathcal{Z} \rightarrow \mathcal{B}$ is compact;
- (b) $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}_0$ is compact;
- (c)

$$\lim_{|z| \rightarrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)| = 0, \quad (30)$$

and

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0. \quad (31)$$

Proof: (a) \Rightarrow (b). It is clear.

(b) \Rightarrow (c). Assume that $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}_0$ is compact, then $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}_0$ is bounded. From the proof of Theorem 4, we known that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re g(z)| = 0,$$

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re \varphi(z)| |g(z)| = 0,$$

Hence, if $\|\varphi\| < 1$,

$$\lim_{|z| \rightarrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)| \leq \frac{1}{1 - \|\varphi\|_\infty} \lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re g(z)| = 0,$$

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} \leq \ln \frac{e}{1 - \|\varphi\|_\infty^2} \lim_{|z| \rightarrow 1} (1 - |z|^2) |\Re g(z)| = 0.$$

from which the result follows in this case.

Assume $\|\varphi\| = 1$. Let $(\varphi(z^k))_{k \in \mathbb{N}}$ be a sequence such that $|\varphi(z^k)| \rightarrow 1$ as $k \rightarrow \infty$. Since $\Re I_\varphi^g : \mathcal{Z}_0 \rightarrow \mathcal{B}$ is compact, by Theorem 2,

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)| = 0, \quad (32)$$

and

$$\lim_{|\varphi(z)| \rightarrow 1} (1 - |z|^2) |\Re g(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0. \quad (33)$$

It is not difficult to see that (28),(32) implies(30). Similar, (27) and (33) imply (31).

(c) \Rightarrow (a). Let $f \in \mathcal{Z}$, we have

$$(1 - |z|^2) |\Re^2(I_\varphi^g(f))(z)| \leq \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\Re \varphi(z)| |g(z)| + (1 - |z|^2) \ln \frac{e}{1 - |\varphi(z)|^2} |\Re g(z)| \right) \|f\|_{\mathcal{Z}}.$$

Taking the supremum in this inequality over all $f \in \mathcal{Z}$ such that $\|f\|_{\mathcal{Z}} \leq 1$.

Letting $|z| \rightarrow 1$ and using (30),(31)

$$\lim_{\|z\| \rightarrow 1} \sup_{\|f\|_{\mathcal{Z}} \leq 1} (1 - |z|^2) |\Re^2(I_\varphi^g(f))(z)| = 0.$$

Using Lemma 3, we obtain that the operator $\Re I_\varphi^g : \mathcal{Z} \rightarrow \mathcal{B}_0$ is compact.

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